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Geometry of paraquaternionic Kähler manifolds with torsion

Simeon Zamkovoy

University of Sofia "St. Kl. Ohridski", Faculty of Mathematics and Informatics, Blvd. James Bourchier 5, 1164 Sofia, Bulgaria

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Abstract

We study the geometry of PQKT connections. We find conditions for the existence of a PQKT connection and prove that if it exists it is unique. We show that PQKT geometry persists in a conformal class of metrics.

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1. Introduction and statement of the results

The geometries of locally supersymmetric vector multiplets in dimensions $(1, 3)$ are known as projective special Kähler geometries. As indicated by the name, such manifolds can be obtained from affine special Kähler manifolds (with homogeneity properties) by a projectivization. This can also be understood from the physical point of view in terms of the conformal calculus. One first constructs a superconformally invariant theory and then eliminates conformal compensators by imposing gauge conditions. This gauge fixing amounts to the projectivization of the scalar manifold underlying the superconformal theory [37,34,11–13,5,4,16,19,1,14]. On the basis of these results such a construction could be adapted to the case of Euclidean signature, and projective special para-Kähler manifolds are constructed in [14]. In Minkowski signature the coupling to supergravity implies that the scalar geometry is quaternionic Kähler instead of hyper-Kähler manifolds [6]. The relation between these two kinds of geometries can again be understood as projectivization, because every quaternionic Kähler manifold can be obtained as the quotient of a hyper-Kähler cone [35]. The identification of the scalar geometry of Euclidean hypermultiplets in rigidly supersymmetric theories is performed in [15]. The fact, that the scalar manifolds of Euclidean hypermultiplets are hyper-para-Kähler manifolds is one of the main results in [15].

We recall that an almost hyper-paracomplex structure on a $4n$-dimensional manifold M is a triple $H = (J_\alpha), \alpha = 1, 2, 3$, of two almost paracomplex structures and one complex structure $J_\alpha : TM \to TM$ satisfying the paraquaternionic identities

$$J_\alpha^2 = \epsilon_\alpha, \quad J_\alpha J_\beta = -J_\beta J_\alpha = -\epsilon_\gamma J_\gamma, \quad \alpha, \beta, \gamma = 1, 2, 3, \epsilon_1 = \epsilon_2 = -\epsilon_3 = 1.$$

E-mail address: zamkovoy@fmi.uni-sofia.bg.

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Here and henceforth (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

When each J_α is an integrable almost (para)complex structure, H is said to be a hyper-paracomplex structure on M. Such a structure is also sometimes called a pseudo-hyper-complex [17]. Any hyper-paracomplex structure admits a unique torsion-free connection ∇^{CP} preserving J_1, J_1, J_3 [2,3] called the complex product connection. Examples of hyper-paracomplex structures on the simple Lie groups $SL(2n+1, \mathbb{R})$, $SU(n, n+1)$ are constructed in [28].

Almost paraquaternionic structures were introduced by Libermann (Libermann called them almost quaternionic structures of the second kind) [33]. An almost paraquaternionic structure on M is a rank-3 subbundle $\mathbb{P} \subset End(TM)$ which is locally spanned by an almost hyper-paracomplex structure $H = (J_\alpha)$; such a locally defined triple H will be called an admissible basis of $\mathbb{P}$. A linear connection ∇ on TM is called paraquaternionic if ∇ preserves $\mathbb{P}$, i.e. $\nabla_X \sigma \in \Gamma(\mathbb{P})$ for all vector fields X and smooth sections $\sigma \in \Gamma(\mathbb{P})$. An almost paraquaternionic structure is said to be paraquaternionic if there is a torsion-free paraquaternionic connection. A $\mathbb{P}$-Hermitian metric is a pseudo-Riemannian metric which is compatible with $\mathbb{P}$ in the sense that $g(J_\alpha X, J_\alpha Y) = -\epsilon_\alpha g(X, Y)$, for $\alpha = 1, 2, 3$. The signature of g is necessarily of neutral type $(2n, 2n)$. An almost paraquaternionic (resp. paraquaternionic) manifold with a $\mathbb{P}$-Hermitian metric is said to be an almost paraquaternionic Hermitian (resp. paraquaternionic Hermitian) manifold.

For $n \geq 2$, the existence of a torsion-free paraquaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated $GL(n, H)Sp(1, \mathbb{R})$-structure [2,3]. The paraquaternionic condition controls the Nijenhuis tensor in the sense that $N(X, Y)J_\alpha := N_\alpha(X, Y)$ preserves the subbundle $\mathbb{P}$. An invariant first-order differential operator D is defined on any almost paraquaternionic manifold which is two-step nilpotent, i.e. $D^2 = 0$ exactly when the structure is paraquaternionic [29]. If the Levi-Civita connection of a paraquaternionic Hermitian manifold $(M, g, \mathbb{P})$ is a paraquaternionic connection then $(M, g, \mathbb{P})$ is called paraquaternionic Kähler (for short, PQK). This condition is equivalent to the statement that the holonomy group of g is contained in $Sp(n, \mathbb{R}).Sp(1, \mathbb{R})$ [28]. A typical example is provided by the paraquaternionic projective space endowed with the standard paraquaternionic Kähler structure [7]. Any paraquaternionic Kähler manifold of dimension $4n \geq 8$ is known to be Einstein [21,36]. If on a PQK manifold there exists a global admissible basis (H) such that each almost (para)complex structure $(J_\alpha) \in (H), \alpha = 1, 2, 3$, is parallel with respect to the Levi-Civita connection then the manifold is called hyper-para-Kähler (for short, HPK). Such manifolds are also called hypersymplectic [23], neutral hyper-Kähler [18,31]. In this case the holonomy group of g is contained in $Sp(n, \mathbb{R}), n > 1$ [36]. Twistor and reflector spaces on paraquaternionic Kähler manifolds are constructed and the integrability of the associated (para)complex structures is investigated in [8] and [24], respectively. These constructions work also in the paraquaternionic case [30].

A natural generalization of PQK spaces is the notion of paraquaternionic Kähler manifolds with torsion (for short, PQKT) which means that there exists a paraquaternionic connection preserving the metric g with totally skew-symmetric torsion of type $(1, 2) + (2, 1)$ with respect to each J_α. More generally, if one considers the same construction in the general case of an almost paraquaternionic structure and defines the almost complex structure on the twistor space (resp. almost paracomplex structure on the reflector space) using horizontal spaces of an arbitrary paraquaternionic connection then the integrability condition is equivalent to the condition that the torsion is of type $(0, 2)$ with respect each J_α [27]. The main objects of interest in this article are the differential geometric properties of PQKT manifolds.

In Section 2 we find necessary and sufficient conditions for the almost paraquaternionic structure to be paraquaternionic structure if the dimension is at least 8 (Theorem 2.6).

In Section 3 we find necessary and sufficient conditions for the existence of a PQKT connection in terms of the Kähler 2-forms and show that the PQKT connection is unique if the dimension is at least 8 (Theorem 3.5), and we prove that the PQKT manifolds are invariant under conformal transformations of the metric.

In Section 4 we prove that the $(2, 0) + (0, 2)$ parts of the Ricci forms ρ_α, ρ_β with respect to J_γ coincide if the dimension is at least 8 (Theorem 4.4). We define torsion 1-form t as a suitable trace of the torsion 3-form and show that ρ_α is of type $(1, 1)$ with respect to J_α if and only if dt is of type $(1, 1)$ with respect to each J_α, $\alpha = 1, 2, 3$, provided the dimension is at least 8 (Theorem 4.7). We show that $\star$-Ricci tensor $\rho^\star_\alpha$ is symmetric if and only if dt is of type $(1, 1)$ with respect to each J_α, $\alpha = 1, 2, 3$ (Corollary 4.8).

In Section 5 we show that there are no homogeneous proper PQKT manifolds (i.e. a homogeneous PQKT which is not PQK or HPKT) with dT of type $(2, 2)$ provided that the torsion is parallel and the dimension is at least 8 (Theorem 5.3).

2. Almost paraquaternionic structures

In this section we study the integrability of almost paraquaternionic structures. We recall that an almost paraquaternionic structure is a G-structure with structure group $GL(n, H).Sp(1, \mathbb{R}) \cong GL(2n, \mathbb{R}).Sp(1, \mathbb{R})$. The almost paraquaternionic structures have been studied intensively in the last few years, especially in the hyper-para-Kähler and paraquaternionic Kähler cases [21,36,7,9,10,30], i.e. the structure group is further reduced to $Sp(n, \mathbb{R})$ or $Sp(n, \mathbb{R}).Sp(1, \mathbb{R})$.

Let ∇ be a paraquaternionic connection, i.e.

$$\nabla J_\alpha = \omega_\beta \otimes J_\gamma + \epsilon_\gamma \omega_\gamma \otimes J_\beta, \tag{2.1}$$

where ω_α, $\alpha = 1, 2, 3$, are 1-forms.

Here and henceforth (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

The Nijenhuis tensor N_α of an almost (para)complex structure J_α is given by

$$N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] + \epsilon_\alpha[X, Y] - J_\alpha[J_\alpha X, Y] - J_\alpha[X, J_\alpha Y].$$

It is well known that an almost (para)complex structure is a (para)complex structure if and only if its Nijenhuis tensor vanishes.

The Nijenhuis bracket $[[A, B]]$ of two endomorphisms is defined in terms of the Lie bracket of vector fields in the following way:

$$\begin{aligned}[[A, B]](X, Y) = {} & [AX, BY] - A[BX, Y] - B[X, AY] + [BX, AY] - B[AX, Y] - A[X, BY] \\ & + (AB + BA)[X, Y]\end{aligned}$$

and $[[J_\alpha, J_\alpha]](X, Y) = 2N_\alpha(X, Y)$.

Proposition 2.1. *Let $H = (J_\alpha)$ be an almost hyper-paracomplex structure on M.*

(1) *There exists a connection ∇^{CP} which preserves H. The connection ∇^{CP} is given by*

$$\begin{aligned}\nabla^{CP}_X Y = {} & \frac{1}{12}\left(\sum_{(\alpha,\beta,\gamma)}(J_\alpha[J_\beta X, J_\gamma Y] - J_\alpha[J_\gamma X, J_\beta Y]) - 2\sum_{\alpha=1}^{3}(\epsilon_\alpha J_\alpha[J_\alpha X, Y] - \epsilon_\alpha J_\alpha[X, J_\alpha Y])\right) \\ & - \frac{1}{12}\sum_{\alpha=1}^{3}(\epsilon_\alpha[J_\alpha X, J_\alpha Y] - \epsilon_\alpha J_\alpha[J_\alpha X, Y] - \epsilon_\alpha J_\alpha[X, J_\alpha Y] + [X, Y]) + \frac{1}{2}[X, Y],\end{aligned} \tag{2.2}$$

where (α, β, γ) indicates the sum over cyclic permutations of $(1, 2, 3)$.

(2) *Its torsion tensor T^H satisfies*

$$T^H = -\frac{1}{12}\sum_{\alpha=1}^{3}\epsilon_\alpha[[J_\alpha, J_\alpha]], \tag{2.3}$$

for any two vector fields X, Y.

Proof. One verifies easily that the formula (2.2) defines a connection ∇^{CP} which preserves H and whose torsion tensor is given by (2.3). □

Remark 2.2. If we define

$$\nabla^0_X Y = \frac{1}{12}\left(\sum_{(\alpha,\beta,\gamma)}(J_\alpha[J_\beta X, J_\gamma Y] - J_\alpha[J_\gamma X, J_\beta Y]) - 2\sum_{\alpha=1}^{3}(\epsilon_\alpha J_\alpha[J_\alpha X, Y] - \epsilon_\alpha J_\alpha[X, J_\alpha Y])\right) + \frac{1}{2}[X, Y],$$

then the connection ∇^0 is characterized by the following properties:

(1) it is torsion-free: $T^{\nabla^0} = 0$;
(2) $\nabla^0 J_\alpha = -\frac{1}{2}[T^H, J_\alpha]$.

The ∇^{CP} connection can be written as

$$\nabla^{CP} = \nabla^0 + \frac{1}{2} T^H.$$

For the case of hyper-paracomplex structure the connection ∇^{CP} was defined in [3].

We need the following

Lemma 2.3. *Let $H = (J_\alpha)$ be an almost hyper-paracomplex structure on M. Then for any vector X, Y the following formulas hold:*

$$\begin{aligned}[[J_\alpha, J_\beta]](X, Y) = {} & J_\alpha T^H(X, J_\beta Y) + J_\alpha T^H(J_\beta X, Y) + J_\beta T^H(X, J_\alpha Y) + J_\beta T^H(J_\alpha X, Y) \\ & - T^H(J_\alpha X, J_\beta Y) - T^H(J_\beta X, J_\alpha Y);\end{aligned} \tag{2.4}$$

$$\begin{aligned}-12\epsilon_\gamma T^H(X, Y) = {} & J_\alpha[[J_\alpha, J_\beta]](X, J_\beta Y) + J_\alpha[[J_\alpha, J_\beta]](J_\beta X, Y) + J_\beta[[J_\alpha, J_\beta]](X, J_\alpha Y) \\ & + J_\beta[[J_\alpha, J_\beta]](J_\alpha X, Y) - [[J_\alpha, J_\beta]](J_\alpha X, J_\beta Y) - [[J_\alpha, J_\beta]](J_\beta X, J_\alpha Y);\end{aligned} \tag{2.5}$$

$$\frac{1}{2}[[J_\alpha, J_\alpha]](X, Y) = -\epsilon_\alpha T^H(X, Y) + J_\alpha T^H(X, J_\alpha Y) + J_\alpha T^H(J_\alpha X, Y) - T^H(J_\alpha X, J_\alpha Y); \tag{2.6}$$

$$\begin{aligned}2[[J_\alpha, J_\alpha]](X, Y) = {} & [[J_\beta, J_\beta]](J_\gamma X, J_\gamma Y) - J_\gamma[[J_\beta, J_\beta]](J_\gamma X, Y) - J_\gamma[[J_\beta, J_\beta]](X, J_\gamma Y) \\ & - \epsilon_\gamma[[J_\beta, J_\beta]](X, Y) + [[J_\gamma, J_\gamma]](J_\beta X, J_\beta Y) - J_\beta[[J_\gamma, J_\gamma]](J_\beta X, Y) \\ & - J_\beta[[J_\gamma, J_\gamma]](X, J_\beta Y) - \epsilon_\beta[[J_\gamma, J_\gamma]](X, Y).\end{aligned} \tag{2.7}$$

Proof. The first three of these equalities follow by definition with long but standard computation. The fourth equality is Proposition 6.1 in [30]. □

As an application of these formulas we obtain necessary and sufficient conditions for an almost hyper-paracomplex structure H to be a hyper-paracomplex structure.

Proposition 2.4. *Let $H = (J_\alpha)$ be an almost hyper-paracomplex structure on M. Then the following conditions are equivalent:*

(1) *H is a hyper-paracomplex structure;*
(2) *two of the almost (para)complex structures J_α $(\alpha = 1, 2, 3)$ are integrable;*
(3) *one of the Nijenhuis brackets $[[J_\alpha, J_\beta]]$ $(\alpha \neq \beta)$ is zero.*

If one of these conditions is verified all Nijenhuis brackets $[[J_\alpha, J_\beta]]$, $\forall\alpha, \beta$, vanish.

Proof. If H is hyper-paracomplex then $T^H = 0$ and (2), (3) follow by (2.6) and (2.4) respectively; vice versa, (2) or (3) imply (1) by (2.7) and (2.5) respectively. □

Proposition 2.5. *Let $\mathbb{P}$ be an almost paraquaternionic structure and $H = (J_\alpha)$ be an admissible basis of $\mathbb{P}$. Let ∇ be a globally defined connection which preserves $\mathbb{P}$ and let T be its torsion tensor. Then*

(1) *There exists a globally defined connection $\nabla^{\mathbb{P}}$ which preserves $\mathbb{P}$. The connection $\nabla^{\mathbb{P}}$ is given by*

$$\begin{aligned}\nabla^{\mathbb{P}}_X Y = {} & \nabla_X Y + \sum_{\alpha=1}^{3}\left(\epsilon_\alpha b_\alpha - \frac{1}{3}\epsilon_\alpha b \circ J_\alpha\right)(X) J_\alpha Y \\ & - \frac{1}{12}\sum_{\alpha=1}^{3}(T(X, Y) - \epsilon_\alpha T(J_\alpha X, J_\alpha Y) + \epsilon_\alpha J_\alpha T(X, J_\alpha Y) + \epsilon_\alpha J_\alpha T(J_\alpha X, Y)),\end{aligned} \tag{2.8}$$

where b_α, b are local 1-forms defined by

$$b_\alpha(X) = \frac{1}{2n-1} tr(J_\alpha T(X)) = -\frac{1}{2n-1}\sum_{i=1}^{4n}\epsilon_i g(T(X, e_i), J_\alpha e_i), b = \sum_{\alpha=1}^{3}\epsilon_\alpha b_\alpha \circ J_\alpha, \quad \alpha = 1, 2, 3;$$

(2) *The torsion tensor* $T^{\mathbb{P}}$ *of* $\nabla^{\mathbb{P}}$ *is given by*

$$T^{\mathbb{P}} = T^H + \partial(C^H), \tag{2.9}$$

where $C^H = \sum_{\alpha=1}^{3} \epsilon_\alpha a_\alpha^H \otimes J_\alpha$, ∂ *denotes the operator of alternation and*

$$a_\alpha^H(X) = \frac{1}{2n-1} tr(J_\alpha T^H(X)) = -\frac{1}{2n-1}\sum_{i=1}^{4n} \epsilon_i g(T^H(X, e_i), J_\alpha e_i), \quad \alpha = 1, 2, 3$$

are the structure 1-forms of H.

Moreover

$$\sum_{\alpha=1}^{3} \epsilon_\alpha a_\alpha^H \circ J_\alpha = 0. \tag{2.10}$$

Proof. For any connection ∇ with torsion tensor T preserving $\mathbb{P}$ we have

$$\begin{aligned}[][J_\alpha, J_\alpha]](X,Y) = {} & 2\partial((\epsilon_\gamma \omega_\gamma \circ J_\alpha - \epsilon_\beta \omega_\beta) \otimes J_\beta + (\omega_\beta \circ J_\alpha + \omega_\gamma) \otimes J_\gamma)(X, Y) \\ & - 2T(J_\alpha X, J_\alpha Y) + 2J_\alpha T(J_\alpha X, Y) + 2J_\alpha T(X, J_\alpha Y) - 2\epsilon_\alpha T(X, Y).\end{aligned}$$

Thus, we have

$$\begin{aligned} & 6T^H(X,Y) + \partial \sum_{(\alpha,\beta,\gamma)} ((2\epsilon_\beta \omega_\alpha + \epsilon_\beta \omega_\gamma \circ J_\beta - \epsilon_\alpha \omega_\beta \circ J_\gamma) \otimes J_\alpha)(X, Y) \\ & \quad = \sum_{\alpha=1}^{3} (T(X,Y) + \epsilon_\alpha T(J_\alpha X, J_\alpha Y) - \epsilon_\alpha J_\alpha T(J_\alpha X, Y) - \epsilon_\alpha J_\alpha T(X, J_\alpha Y)). \end{aligned} \tag{2.11}$$

One verifies easily that the formula (2.8) defines a connection $\nabla^{\mathbb{P}}$ which preserves $\mathbb{P}$ and whose torsion tensor is

$$\begin{aligned} 6T^{\mathbb{P}}(X,Y) = {} & \sum_{\alpha=1}^{3} (T(X,Y) + \epsilon_\alpha T(J_\alpha X, J_\alpha Y) - \epsilon_\alpha J_\alpha T(J_\alpha X, Y) - \epsilon_\alpha J_\alpha T(X, J_\alpha Y)) \\ & + 6\partial \sum_{\alpha=1}^{3} \left(\left(\epsilon_\alpha b_\alpha - \frac{1}{3} b \circ J_\alpha \right) \otimes J_\alpha \right)(X, Y). \end{aligned} \tag{2.12}$$

Taking the appropriate trace in (2.11), we get

$$3\epsilon_\alpha a_\alpha^H = (2\epsilon_\beta \omega_\alpha + \epsilon_\beta \omega_\gamma \circ J_\beta - \epsilon_\alpha \omega_\beta \circ J_\gamma) + (2\epsilon_\alpha b_\alpha + b_\beta \circ J_\gamma - b_\gamma \circ J_\beta). \tag{2.13}$$

Now, equalities (2.11)–(2.13) prove (2.9). The equality (2.10) follows from (2.13). □

The following theorem gives the necessary and sufficient condition for an almost paraquaternionic structure to be a paraquaternionic structure.

Theorem 2.6. *An almost paraquaternionic structure* $\mathbb{P}$ *is a paraquaternionic structure if and only if* $T^{\mathbb{P}} = 0$.

Proof. (1) Assume $T^{\mathbb{P}} = 0$. Then T^H has the form

$$T^H = -\partial \sum_{\alpha=1}^{3} \epsilon_\alpha a_\alpha^H \otimes J_\alpha.$$

It is to easy check that $\nabla = \nabla^H + \sum_{\alpha=1}^{3} \epsilon_\alpha a_\alpha^H \otimes J_\alpha$ is a torsion-free connection. From equalities $\nabla J_\alpha = \bar{\omega_\beta} \otimes J_\gamma + \epsilon_\gamma \bar{\omega_\gamma} \otimes J_\beta$, where $\bar{\omega_\alpha} = \omega_\alpha - 2\epsilon_\gamma a_\alpha^H$, and it follows that ∇ preserves $\mathbb{P}$.

(2) Now let ∇ be a paraquaternionic connection and $H = (J_\alpha)$ an admissible basis of $\mathbb{P}$. For any torsion-free connection $\bar{\nabla}$ preserving $\mathbb{P}$ we have

$$[[J_\alpha, J_\alpha]] = 2\partial((\epsilon_\gamma \omega_\gamma \circ J_\alpha - \epsilon_\beta \omega_\beta) \otimes J_\beta + (\omega_\beta \circ J_\alpha + \omega_\gamma) \otimes J_\gamma).$$

Thus we obtain

$$6T^H = -\partial \sum_{(\alpha,\beta,\gamma)} (2\epsilon_\beta\omega_\alpha + \epsilon_\beta\omega_\gamma \circ J_\beta - \epsilon_\alpha\omega_\beta \circ J_\gamma) \otimes J_\alpha. \tag{2.14}$$

From the formula (2.13), we get $6T^H = -6\partial \sum_{\alpha=1}^3 \epsilon_\alpha a_\alpha^H \otimes J_\alpha$. Hence $T^{\mathbb{P}} = 0$. □

3. Characterizations of the PQKT connection

Let $(M, g, (J_\alpha) \in \mathbb{P}, \alpha = 1, 2, 3)$ be a $4n$-dimensional almost paraquaternionic manifold with $\mathbb{P}$-Hermitian pseudo-Riemannian metric g. We shall work locally with an admissible basis (J_α). The Kähler form F_α of each J_α is defined by $F_\alpha = g(., J_\alpha .)$. The corresponding Lee forms are given by $\theta_\alpha = -\epsilon_\alpha\delta F_\alpha \circ J_\alpha$.

For an r-form ψ we denote by $J_\alpha\psi$ the r-form defined by $J_\alpha\psi(X_1, \ldots, X_r) := (-1)^r\psi(J_\alpha X_1, \ldots, J_\alpha X_r), \alpha = 1, 2, 3$. We shall use the notation $d_\alpha F_\beta(X, Y, Z) = -dF_\beta(J_\alpha X, J_\alpha Y, J_\alpha Z), \alpha, \beta = 1, 2, 3$.

We recall the decomposition of a skew-symmetric tensor $P \in \Lambda^2T^*M \otimes TM$ with respect to a given almost (para)complex structure J_α. The $(1, 1)$, $(2, 0)$ and $(0, 2)$ parts of P are defined by $P^{1,1}(J_\alpha X, J_\alpha Y) = -\epsilon_\alpha P^{1,1}(X, Y)$, $P^{2,0}(J_\alpha X, Y) = J_\alpha P^{2,0}(X, Y)$, $P^{0,2}(J_\alpha X, Y) = -J_\alpha P^{0,2}(X, Y)$, respectively.

For each $\alpha = 1, 2, 3$, we denote by dF_α^+ (resp. dF_α^-) the $(1, 2) + (2, 1)$ part (resp. $(3, 0) + (0, 3)$ part) of dF_α with respect to the almost (para)complex structure J_α. We consider the following 1-forms

$$\theta_{\alpha,\beta} = \epsilon_\alpha \frac{1}{2} \sum_{i=1}^{4n} \epsilon_i dF_\alpha^+(X, e_i, J_\beta e_i), \quad \alpha, \beta = 1, 2, 3.$$

Here and further $e_1, e_2, \ldots, e_n, e_{n+1} = J_3e_1, e_{n+2} = J_3e_2, \ldots, e_{2n} = J_3e_n, e_{2n+1} = J_1e_1, e_{2n+2} = J_1e_2, \ldots, e_{3n} = J_1e_n, e_{3n+1} = J_2e_1, e_{3n+2} = J_2e_2, \ldots, e_{4n} = J_2e_n$ where $J_1e_i = e_{2n+i}, J_1e_{n+i} = e_{3n+i}, J_2e_i = e_{3n+i}, J_2e_{n+i} = -e_{2n+i}, J_3e_{2n+i} = -e_{3n+i}, J_3e_i = e_{2n+i}, i = 1 \ldots n$ and $g(e_i, e_i) = \epsilon_i$ where $\epsilon_i = +1, i = 1, \ldots, 2n$ and $\epsilon_i = -1, i = 2n + 1, \ldots, 4n$ is an orthonormal basis of the tangential space.

Note that $\theta_{\alpha,\alpha} = \theta_\alpha$.

Let $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ be the torsion tensor of ∇. We denote by the same letter the torsion tensor of type (0,3) given by $T(X, Y, Z) = g(T(X, Y), Z)$. The Nijenhuis tensor is expressed in terms of ∇ as follows

$$N_\alpha(X, Y) = -\epsilon_\alpha 4T_\alpha^{0,2}(X, Y) + (\nabla_{J_\alpha X} J_\alpha)(Y) - (\nabla_{J_\alpha Y} J_\alpha)(X) - (\nabla_Y J_\alpha)(J_\alpha X) + (\nabla_X J_\alpha)(J_\alpha Y),$$

where the $(0, 2)$ part $T_\alpha^{0,2}$ of the torsion with respect to J_α is given by

$$T_\alpha^{0,2}(X, Y) = \frac{1}{4}\left(T(X, Y) + \epsilon_\alpha T(J_\alpha X, J_\alpha Y) - \epsilon_\alpha J_\alpha T(J_\alpha X, Y) - \epsilon_\alpha J_\alpha T(X, J_\alpha Y)\right). \tag{3.15}$$

A 3-form ψ is of type $(1, 2) + (2, 1)$ with respect to an almost (para)complex structure J_α if and only if it satisfies the equality [30]

$$-\epsilon_\alpha\psi(X, Y, Z) = \psi(J_\alpha X, J_\alpha Y, Z) + \psi(J_\alpha X, Y, J_\alpha Z) + \psi(X, J_\alpha Y, J_\alpha Z). \tag{3.16}$$

Definition. An almost paraquaternionic Hermitian manifold $(M, g, (H_\alpha) \in \mathbb{P})$ is a *PQKT manifold* if it admits a metric paraquaternionic connection ∇ with totally skew symmetric torsion which is a $(1, 2)+(2, 1)$-form with respect to each $J_\alpha, \alpha = 1, 2, 3$. If the torsion 3-form is closed then the manifold is said to be a *strong PQKT manifold*.

It follows that the holonomy group of ∇ is a subgroup of $Sp(n, \mathbb{R}).Sp(1, \mathbb{R})$.

By means of (2.1), (3.15) and (3.16), the Nijenhuis tensor N_α of $J_\alpha, \alpha = 1, 2, 3$, on a PQKT manifold is given by

$$N_\alpha(X, Y) = -A_\alpha(Y)J_\beta X + A_\alpha(X)J_\beta Y - J_\alpha A_\alpha(Y)J_\gamma X + J_\alpha A_\alpha(X)J_\gamma Y, \tag{3.17}$$

where

$$A_\alpha = \omega_\beta - \epsilon_\alpha J_\alpha\omega_\gamma. \tag{3.18}$$

Remark 1. The torsion of ∇ is a $(1,2)+(2,1)$-form with respect to any (local) almost (para)complex structure $J \in \mathbb{P}$. In fact, it is sufficient that the torsion is a $(1,2)+(2,1)$-form with respect to the only two almost (para)complex structures of (H) since Proposition 6.1 in [30] gives the necessary expression for N_{J_α} by N_{J_β} and N_{J_γ}. Indeed, it is easy to see that the formula from Proposition 6.1 in [30] holds for the $(0,2)$ part $T_\alpha^{0,2}$, $\alpha = 1,2,3$, of the torsion. Hence, the vanishing of the $(0,2)$ part of the torsion with respect to any two almost (para)complex structures in (H) implies the vanishing of the $(0,2)$ part of T with respect to the third one.

On a PQKT manifold there are three naturally associated 1-forms defined by

$$t_\alpha(X) = \epsilon_\alpha \frac{1}{2} \sum_{i=1}^{4n} \epsilon_i T(X, e_i, J_\alpha e_i), \quad \alpha = 1,2,3. \tag{3.19}$$

Following [25], we have

Proposition 3.1. *On a PQKT manifold* $J_1 t_1 = J_2 t_2 = J_3 t_3$.

Proof. Applying (3.16) with respect to J_β we obtain

$$\begin{aligned} t_\alpha(X) &= \epsilon_\alpha \frac{1}{2} \sum_{i=1}^{4n} \epsilon_i T(X, e_i, J_\alpha e_i) = -\frac{1}{2} \sum_{i=1}^{4n} T(X, J_\beta e_i, J_\gamma e_i) \\ &= \frac{1}{2} \sum_{i=1}^{4n} \epsilon_i T(J_\beta X, e_i, J_\gamma e_i) + \frac{1}{2} \sum_{i=1}^{4n} \epsilon_i T(J_\beta X, e_i, J_\gamma e_i) - \epsilon_\alpha \frac{1}{2} \sum_{i=1}^{4n} \epsilon_i T(X, e_i, J_\alpha e_i). \end{aligned}$$

The last equality implies $t_\alpha = \epsilon_\alpha \epsilon_\beta J_\beta t_\gamma$ which proves the assertion. □

We introduce the torsion 1-form on PQKT manifolds via the equality

$$t(X) = -\frac{1}{2} \sum_{i=1}^{4n} \epsilon_\alpha \epsilon_i T(J_\alpha X, e_i, J_\alpha e_i). \tag{3.20}$$

We need the following

Lemma 3.2. *For a 3-form T of type $(1,2)+(2,1)$ with respect to each J_α one has*

$$\sum_{i,j=1}^{4n} \epsilon_i \epsilon_j g(T(e_i, e_j), T(J_\gamma e_i, J_\beta e_j)) = 0, \qquad \sum_{i,j=1}^{4n} \epsilon_i \epsilon_j g(T(e_i, e_j), T(J_\beta e_i, J_\beta e_j)) = -\frac{1}{3} \epsilon_\beta |T|^2,$$

where $|\cdot|^2$ denotes the norm with respect to the metric g.

Proof. This proof is very similar to the proof of Lemma 3.2 in [26] and we omit it. □

Further, we have

Theorem 3.3. *Let $(M, g, (J_\alpha \in \mathbb{P}))$ be a $4n$-dimensional PQKT manifold. Then the following identities hold*

$$\sum_{i,j=1}^{4n} \epsilon_i (\nabla_X T)(J_\alpha Y, e_i, J_\alpha e_i) = -2\epsilon_\alpha (\nabla_X t)(Y); \tag{3.21}$$

$$\sum_{i,j=1}^{4n} \epsilon_i \epsilon_j dT(e_j, J_\alpha e_j, e_i J_\alpha e_i) = 8\epsilon_\alpha \delta t - 8\epsilon_\alpha |t|^2 + \frac{4}{3}\epsilon_\alpha |T|^2, \qquad \sum_{i,j=1}^{4n} \epsilon_i \epsilon_j dT(e_j, J_\beta e_j, e_i J_\gamma e_i) = 0, \tag{3.22}$$

where δ is the codifferential with respect to g.

Proof. The formula (3.21) follows from (2.1) and definition (3.20) of the torsion 1-form by straightforward calculations. To prove (3.22) we need the expression for dT in terms of ∇ [20,25],

$$dT(X,Y,Z,U) = \underset{XYZ}{\sigma}\{(\nabla_X T)(Y,Z,U) + 2g(T(X,Y),T(Z,U))\} - (\nabla_U T)(X,Y,Z), \tag{3.23}$$

where $\underset{XYZ}{\sigma}$ denotes the cyclic sum of X, Y, Z. Taking the appropriate trace in (3.23) and applying Lemma 3.2 we obtain the first equality in (3.22). Finally, from (3.23) combined with (3.21) and Lemma 3.2 we get that

$$\sum_{i,j=1}^{4n} \epsilon_i\epsilon_j dT(e_j, J_\beta e_j, e_i J_\gamma e_i) = -4\sum_{i,j=1}^{4n} \epsilon_i\epsilon_j g(T(e_i,e_j), T(J_\gamma e_i, J_\beta e_j)) = 0. \quad \square$$

Theorem 3.4. *Every PQKT is a paraquaternionic manifold.*

Proof. This is an immediate consequence of (3.17) and Theorem 2.6 $\square$

However, the converse to the above property is not always true. In fact, we have

Theorem 3.5. *Let $(M, g, (J_\alpha \in \mathbb{P}))$ be a $4n$-dimensional $(n > 1)$ paraquaternionic manifold with $\mathbb{P}$-Hermitian metric g. Then M admits a PQKT structure if and only if the following conditions hold*

$$(d_\alpha F_\alpha)^+ - (d_\beta F_\beta)^+ = \frac{1}{2}\left(\epsilon_\gamma K_\alpha \wedge F_\beta - \epsilon_\beta J_\beta K_\beta \wedge F_\alpha - \epsilon_\alpha (K_\beta - J_\alpha K_\alpha)\wedge F_\gamma\right), \tag{3.24}$$

where $(d_\alpha F_\alpha)^+$ denotes the $(1,2)+(2,1)$ part of $(d_\alpha F_\alpha)$ with respect to $J_\alpha, \alpha = 1,2,3,$ and the 1-forms $K_\alpha, \alpha = 1,2,3,$ are given by

$$K_\alpha = \frac{1}{1-n}\left(\epsilon_\alpha J_\beta \theta_\alpha + \epsilon_\beta \theta_{\alpha,\gamma}\right). \tag{3.25}$$

The metric paraquaternionic connection ∇ with torsion 3-form of type $(1,2)+(2,1)$ is unique and is determined by

$$\nabla = \nabla^g + \frac{1}{2}\left((d_\alpha F_\alpha)^+ - \frac{1}{2}\left(\epsilon_\alpha J_\alpha K_\alpha \wedge F_\gamma + \epsilon_\gamma K_\alpha \wedge F_\beta\right)\right), \tag{3.26}$$

where ∇^g is the Levi-Civita connection of g.

Proof. To prove the 'if' part, let ∇ be a metric paraquaternionic connection satisfying (2.1) whose torsion T has the required properties. We follow the scheme in [25]. Since T is skew-symmetric, we have

$$\nabla = \nabla^g + \frac{1}{2}T. \tag{3.27}$$

We obtain using (2.1) and (3.27) that

$$\frac{1}{2}(T(X, J_\alpha Y, Z)) + (T(X,Y,J_\alpha Z)) = -g\left((\nabla^g_X J_\alpha)Y, Z\right) - \omega_\beta(X)F_\gamma(Y,Z) - \epsilon_\gamma\omega_\gamma(X)F_\beta(Y,Z). \tag{3.28}$$

The tensor $\nabla^g J_\alpha$ is decomposed by parts according to $\nabla J_\alpha = (\nabla J_\alpha)^{2,0} + (\nabla J_\alpha)^{0,2}$, where [30,22]

$$g\left((\nabla^g_X J_\alpha)^{2,0}Y, Z\right) = -\frac{1}{2}\left(\epsilon_\alpha (dF_\alpha)^+(X, J_\alpha Y, J_\alpha Z) + (dF_\alpha)^+(X,Y,Z)\right) \tag{3.29}$$

$$g\left((\nabla^g_X J_\alpha)^{0,2}Y, Z\right) = \frac{1}{2}\left(g(N_\alpha(X,Y), J_\alpha Z) - g(N_\alpha(X,Z), J_\alpha Y) - g(N_\alpha(Y,Z), J_\alpha X)\right). \tag{3.30}$$

Taking the $(2,0)$ part in (3.28) we obtain using (3.29) that

$$T(X, J_\alpha Y, Z) + T(X, Y, J_\alpha Z) = (\epsilon_\alpha dF_\alpha^+(X, J_\alpha Y, J_\alpha Z) + dF_\alpha^+(X,Y,Z)) - C_\alpha(X)F_\gamma(Y,Z) + \epsilon_\gamma C_\alpha(J_\alpha X)F_\beta(Y,Z), \tag{3.31}$$

where

$$C_\alpha = \omega_\beta + \epsilon_\alpha J_\alpha \omega_\gamma. \tag{3.32}$$

The cyclic sum of (3.31) and the fact that T and $(dF_\alpha)^+$ are $(1,2)+(2,1)$-forms with respect to each J_α give

$$T = (d_\alpha F_\alpha)^+ - \frac{1}{2}\left(\epsilon_\alpha J_\alpha C_\alpha \wedge F_\gamma + \epsilon_\gamma C_\alpha \wedge F_\beta\right). \tag{3.33}$$

Further, we take the contractions in (3.33) to get

$$\begin{aligned} J_\alpha t_\alpha &= -\theta_\alpha - \epsilon_\gamma J_\beta C_\alpha, \\ J_\alpha t_\alpha &= \epsilon_\beta J_\gamma \theta_{\beta,\alpha} - n\epsilon_\alpha J_\gamma C_\beta, \\ J_\alpha t_\alpha &= -\epsilon_\gamma J_\beta \theta_{\gamma,\alpha} - n\epsilon_\beta J_\alpha C_\gamma. \end{aligned} \tag{3.34}$$

Using Proposition 3.1, (3.18) and (3.32), we obtain consequently from (3.34) that

$$\epsilon_\alpha A_\alpha = -J_\alpha C_\beta + \epsilon_\alpha J_\gamma C_\gamma = J_\beta\left(\theta_\gamma - \theta_\beta\right), \tag{3.35}$$

$$(n-1)\epsilon_\gamma J_\beta C_\alpha = \theta_\alpha + \epsilon_\alpha J_\beta \theta_{\alpha,\gamma}. \tag{3.36}$$

Then (3.24) and (3.25) follow from (3.33) and (3.36).

For the converse, we define ∇ by (3.26). To complete the proof we have to show that ∇ is a paraquaternionic connection. We calculate

$$\begin{aligned} g\left((\nabla_X J_\alpha)Y, Z\right) &= g\left((\nabla^g_X J_\alpha)Y, Z\right) + \frac{1}{2}\left(T(X, J_\alpha Y, Z) + T(X, Y, J_\alpha Z)\right) \\ &= -\omega_\beta(X) F_\gamma(Y,Z) - \epsilon_\gamma \omega_\gamma(X) F_\beta(Y,Z), \end{aligned}$$

where we used (3.29), (3.30), (3.35), (3.25), (3.18), (3.32) and the compatibility condition (3.24) to get the last equality. The uniqueness of ∇ follows from (3.26). □

In the case of a hyper-para-Kähler manifold with torsion (for short, HPKT), $K_\alpha = dF_\alpha^- = 0$ and Theorem 3.5 is a consequence of the general results in [30] which imply that on a para-Hermitian manifold there exists a unique linear connection with totally skew-symmetric torsion preserving the metric and the (para)complex structure.

As a consequence of the proof of Theorem 3.5, we get

Proposition 3.6. *The Nijenhuis tensors of a PQKT manifold depend only on the difference between the Lie forms. In particular, the almost (para)complex structures J_α on a PQKT manifold $(M, (J_\alpha) \in \mathbb{P}, g, \nabla)$ are integrable if and only if*

$$\theta_\alpha = \theta_\beta = \theta_\gamma.$$

Proof. The Nijenhuis tensors are given by (3.17) and (3.35). □

Corollary 3.7. *On a $4n$-dimensional PQKT manifold the following formulas hold*

$$\begin{aligned} &J_\beta \theta_{\alpha,\gamma} = -J_\gamma \theta_{\alpha,\beta}, \\ &(n^2+n)\theta_\alpha - n\theta_\beta - n^2\theta_\gamma - \epsilon_\beta J_\gamma \theta_{\beta,\alpha} - n\epsilon_\gamma J_\alpha \theta_{\gamma,\beta} + (n+1)\epsilon_\alpha J_\beta \theta_{\alpha,\gamma} = 0. \end{aligned} \tag{3.37}$$

If $n = 1$ then

$$\theta_\alpha = -\epsilon_\alpha J_\beta \theta_{\alpha,\gamma} = \epsilon_\alpha J_\gamma \theta_{\alpha,\beta}.$$

Proof. The first formula follows directly from the system (3.34). Solving the system (3.34) with respect to C_α we obtain

$$(n^3-1)\epsilon_\gamma J_\beta C_\alpha = (\theta_\alpha + \epsilon_\beta J_\gamma \theta_{\beta,\alpha}) + n(\theta_\beta + \epsilon_\gamma J_\alpha \theta_{\gamma,\beta}) + n^2(\theta_\gamma + \epsilon_\alpha J_\beta \theta_{\alpha,\gamma}). \tag{3.38}$$

Then (3.37) is a consequence of (3.38) and (3.36). The last assertion follows from (3.36). □

Corollary 3.8. *On a* $4n$*-dimensional* ($n > 1$) *PQKT manifold the* $sp(1, \mathbb{R})$ *connection 1-forms are given by*

$$\omega_\beta = \frac{1}{2}\epsilon_\alpha J_\beta \left(\theta_\gamma - \theta_\beta + \frac{1}{1-n}\theta_\alpha\right) + \frac{1}{2(1-n)}\epsilon_\beta \theta_{\alpha,\gamma}. \tag{3.39}$$

Proof. The proof follows in a straightforward way from (3.35), (3.36), (3.18) and (3.32). □

Theorem 3.5 and the above formulas lead to the following criterion.

Proposition 3.9. *Let* $(M, g, (J_\alpha))$ *be a* $4n$*-dimensional* ($n > 1$) *PQKT manifold. The following conditions are equivalent:*

(i) $(M, g, (H))$ *is a local HPKT manifold;*
(ii) $d_\alpha F_\alpha^+ = d_\beta F_\beta^+ = d_\gamma F_\gamma^+$*;*
(iii) $\theta_\alpha = -\epsilon_\alpha J_\beta \theta_{\alpha,\gamma}$.

Proof. If $(M, g, (H))$ is an HPKT manifold, the connection 1-forms are $\omega_\alpha = 0, \alpha = 1, 2, 3$. Then (ii) and (iii) follow from (3.32), (3.36), (3.25) and (3.24).

If (iii) holds, then (3.36) and (3.35) yield $C_\alpha = A_\alpha = 0, \alpha = 1, 2, 3$, since $n > 1$. Consequently, $2\omega_\alpha = J_\beta C_\beta - J_\beta A_\beta = 0$ by (3.32) and (3.18). Thus the equivalence of (i) and (iii) is proved.

Let (ii) holds. Then we compute that $\theta_\alpha = J_\gamma \theta_{\beta,\alpha}$. Since $n > 1$, the equality (3.38) leads to $C_\alpha = 0, \alpha = 1, 2, 3$, which forces $\omega_\alpha = 0, \alpha = 1, 2, 3$, as above. This completes the proof. □

The next theorem shows that PQKT manifolds are stable under conformal transformations.

Theorem 3.10. *Let* $(M, g, (J_\alpha), \nabla)$ *be a* $4n$*-dimensional PQKT manifold. Then every pseudo-Riemannian metric* $\bar{g}$ *in the conformal class* $[g]$ *admits a PQKT connection. If* $\bar{g} = fg$ *for a function* f *then the PQKT connection* $\bar{\nabla}$ *corresponding to* $\bar{g}$ *is given by*

$$\begin{aligned}\bar{g}(\bar{\nabla}_X Y, Z) = {} & fg(\nabla_X Y, Z) + \frac{1}{2}(df(X)g(Y,Z) + df(Y)g(X,Z) - df(Z)g(X,Y)) \\ & - \frac{1}{2}\left(\epsilon_\alpha J_\alpha df \wedge F_\alpha + \epsilon_\beta J_\beta df \wedge F_\beta + \epsilon_\gamma J_\gamma df \wedge F_\gamma\right)(X,Y,Z).\end{aligned} \tag{3.40}$$

The torsion tensors T *and* $\bar{T}$ *and the torsion 1-forms* t *and* $\bar{t}$ *of* ∇ *and* $\bar{\nabla}$ *are related by*

$$\bar{T} = fT - \epsilon_\alpha J_\alpha df \wedge F_\alpha - \epsilon_\beta J_\beta df \wedge F_\beta - \epsilon_\gamma J_\gamma df \wedge F_\gamma. \tag{3.41}$$

$$\bar{t} = t - (2n+1)d \ln f. \tag{3.42}$$

Proof. First we assume $n > 1$. We apply Theorem 3.5 to the paraquaternionic Hermitian manifold $(M, \bar{g} = fg, (J_\alpha) \in \mathbb{P})$. We denote the objects corresponding to the metric $\bar{g}$ by a line above the symbol, e.g. $\bar{F}_\alpha$ denotes the Kähler form of J_α with respect to $\bar{g}$. An easy calculation gives the following sequence of formulas

$$d_\alpha \bar{F}_\alpha^+ = -\epsilon_\alpha J_\alpha df \wedge F_\alpha + f d_\alpha F_\alpha^+; \quad \bar{\theta}_\alpha = \theta_\alpha + (2n-1)d \ln f; \quad \bar{\theta}_{\alpha,\gamma} = \theta_{\alpha,\gamma} + \epsilon_\gamma J_\beta d \ln f. \tag{3.43}$$

We substitute (3.43) into (3.25), (3.35) and (3.39) to get

$$\bar{K}_\alpha = K_\alpha - 2\epsilon_\alpha J_\beta d \ln f, \quad \bar{A} = A, \quad \bar{\omega}_\alpha = \omega_\alpha - \epsilon_\alpha J_\beta d \ln f. \tag{3.44}$$

Using (3.43) and (3.44) we verify that the conditions (3.24) with respect to the metric $\bar{g}$ are fulfilled. Theorem 3.5 implies that there exists a PQKT connection $\bar{\nabla}$ with respect to $(\bar{g}, P)$. Using the well known relation between the Levi-Civita connections of conformally equivalent metrics, (3.43) and (3.44), we obtain (3.40) from (3.26).

Using (3.40), we get (3.41) and consequently (3.42). □

Namely, any conformal metric of a PQK, HPK or HPKT manifold will give a PQKT manifold. This leads to the notion of *locally conformally PQK (resp. locally conformally HPK, resp. locally conformally HPKT) manifolds* (for short, l.c.PQK (resp. l.c.HPK, resp. l.c.HPKT) manifolds) in the context of PQKT geometry.

We recall that a paraquaternionic Hermitian manifold $(M, g, \mathbb{P})$ is said to be an l.c.PQK (resp. l.c.HPK, resp. l.c.HPKT) manifold if each point $p \in M$ has a neighborhood U_p such that $g|_{U_p}$ is conformally equivalent to a PQK (resp. HPK, resp. HPKT) metric.

For example, the Kodaira–Thurston surface modeled on $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}/\Gamma$ is on example of a compact l.c.HPKT which is not globally conformal HPKT [30].

Theorems 3.10, 3.5 together with Propositions 3.6 and 3.9 imply the following

Corollary 3.11. *Every l.c.PQK manifold admits a PQKT structure.*

Further, if $(M, g, (J_\alpha), \nabla)$ is a $4n$-dimensional $n > 1$ PQKT manifold then:

(i) $(M, g, (J_\alpha), \nabla)$ *is an l.c.PQK manifold if and only if*

$$T = -\frac{1}{2n+1}\left(t_\alpha \wedge F_\alpha + t_\beta \wedge F_\beta + t_\gamma \wedge F_\gamma\right), \quad dt = 0; \tag{3.45}$$

(ii) $(M, g, (J_\alpha), \nabla)$ *is an l.c.HPKT manifold if and only if the 1-form $\theta_\alpha + \epsilon_\alpha J_\beta \theta_{\alpha,\gamma}$ is closed, i.e.*

$$d(\theta_\alpha + \epsilon_\alpha J_\beta \theta_{\alpha,\gamma}) = 0;$$

(iii) $(M, g, (J_\alpha), \nabla)$ *is an l.c.HPK manifold if an only if (3.45) holds and*

$$\theta_\alpha + \epsilon_\alpha J_\beta \theta_{\alpha,\gamma} = \frac{2(1-n)}{2n+1} t.$$

4. Curvature of a PQKT space

Let $R = [\nabla, \nabla] - \nabla_{[,]}$ be the curvature tensor of type (1,3) of ∇. We denote the curvature tensor of type (0,4) $R(X, Y, Z, V) = g(R(X, Y)Z, V)$ by the same letter. There are three Ricci forms and three scalar functions given by

$$\rho_\alpha(X, Y) = \frac{1}{2}\sum_{i=1}^{4n} \epsilon_i R(X, Y, e_i, J_\alpha e_i), \quad \alpha = 1, 2, 3,$$

$$Scal_{\alpha,\beta} = \sum_{i=1}^{4n} \epsilon_i \epsilon_\alpha \rho_\alpha(e_i, J_\beta e_i), \quad \alpha, \beta = 1, 2, 3.$$

Proposition 4.1. *The curvature of a PQKT manifold $(M, g, (J_\alpha), \nabla)$ satisfies the following relations*

$$[R(X, Y), J_\alpha] = \frac{1}{n}\left(-\epsilon_\alpha \rho_\gamma(X, Y) \otimes J_\beta + \epsilon_\alpha \rho_\beta(X, Y) \otimes J_\gamma\right), \tag{4.46}$$

$$\epsilon_\gamma \rho_\alpha = d\omega_\alpha + \epsilon_\alpha \omega_\beta \wedge \omega_\gamma. \tag{4.47}$$

Proof. We follow the classical scheme (see e.g. [30]). Using (2.1), we obtain

$$[R(X, Y), J_\alpha] = (d\omega_\beta + \epsilon_\beta \omega_\gamma \wedge \omega_\alpha)(X, Y) \otimes J_\gamma + \epsilon_\gamma (d\omega_\gamma + \epsilon_\gamma \omega_\alpha \wedge \omega_\beta)(X, Y) \otimes J_\beta.$$

Taking the trace in the last equality, we get

$$\begin{aligned} \rho_\alpha(X, Y) &= \frac{1}{2}\sum_{i=1}^{4n} \epsilon_i R(X, Y, e_i, J_\alpha e_i) = -\frac{1}{2}\sum_{i=1}^{4n} \epsilon_\beta \epsilon_i R(X, Y, J_\beta e_i, J_\alpha J_\beta e_i) \\ &= -\frac{1}{2}\sum_{i=1}^{4n} \epsilon_i R(X, Y, e_i, J_\alpha e_i) + 2n\epsilon_\gamma (d\omega_\alpha + \epsilon_\alpha \omega_\beta \wedge \omega_\gamma)(X, Y). \quad \square \end{aligned}$$

Using Proposition 4.1 we find a simple necessary and sufficient condition for a PQKT manifold to be an HPKT one, i.e. the holonomy group of ∇ to be a subgroup of $Sp(n, \mathbb{R})$.

Proposition 4.2. *A $4n$-dimensional $(n > 1)$ PQKT manifold is a local HPKT manifold if and only if all the three Ricci forms vanish, i.e. $\rho_1 = \rho_2 = \rho_3 = 0$.*

Proof. If a PQKT manifold is an HPKT manifold then the holonomy group of ∇ is contained in $Sp(n,\mathbb{R})$. This implies $\rho_\alpha = 0, \quad \alpha = 1,2,3$.

For the converse, let the three Ricci forms vanish. Eqs. (4.47) mean that the curvature of the $Sp(1,\mathbb{R})$ connection on $\mathbb{P}$ vanish. Then there exists a local basis $(I_\alpha, \alpha = 1,2,3)$ of almost (para)complex structures on $\mathbb{P}$ and each I_α is ∇ parallel, i.e. the corresponding connection 1-forms $\omega_{I_\alpha} = 0, \alpha = 1,2,3$. Then each I_α is a (para)complex structure, by (3.17) and (3.18). This implies that the PQKT manifold is a local HPKT manifold. □

The Ricci tensor Ric and scalar curvatures $Scal$ and $Scal_\alpha$ of the PQKT connection ∇ are defined by

$$Ric(X,Y) = \sum_{i=1}^{4n} \epsilon_i R(e_i, X, Y, e_i), \quad Scal = \sum_{i=1}^{4n} \epsilon_i Ric(e_i, e_i), \quad Scal_\alpha = -\sum_{i=1}^{4n} \epsilon_i Ric(e_i, J_\alpha e_i).$$

We denote by $Ric^g, Scal^g, \rho^g_\alpha$, etc. the corresponding objects for the metric g, i.e. the same objects taken with respect to the Levi-Civita connection ∇^g. We may consider (g, J_α) as an almost (para)Hermitian structure. The tensor $\rho^\star_\alpha(X,Y) = \rho^g_\alpha(X, J_\alpha Y)$ is known as the $\star$-Ricci tensor of the almost (para)Hermitian structure. It is equal to $\rho^\star_\alpha(X,Y) = -\sum_{i=1}^{2n} R^g(e_i, X, J_\alpha Y, J_\alpha e_i)$ by the Bianchi identity. The function $Scal^g_\alpha$ is known also as the $\star$-scalar curvature. In general, the $\star$-Ricci tensor is not symmetric and the $\star$-Einstein condition is a strong condition. We shall see in this section that the scalar curvature functions are not independent and we define a new scalar invariant, the "paraquaternionic $\star$-scalar curvature" of a PQKT space.

Our main technical result is the following

Proposition 4.3. *Let $(M, g, (J_\alpha), \nabla)$ be a $4n$-dimensional PQKT manifold. The following formulas hold*

$$\begin{aligned} & n\epsilon_\alpha \rho_\alpha(X, J_\alpha Y) + \epsilon_\beta \rho_\beta(X, J_\beta Y) + \epsilon_\gamma \rho_\gamma(X, J_\gamma Y) \\ & \quad = nRic(X,Y) + \frac{n}{4}\epsilon_\alpha (dT)_\alpha(X, J_\alpha Y) - n(\nabla_X t)(Y); \end{aligned} \tag{4.48}$$

$$\begin{aligned} (n-1)\epsilon_\alpha \rho_\alpha(X, J_\alpha Y) \; = \; & \frac{n(n-1)}{n+2} Ric(X,Y) - \frac{n(n-1)}{(n+2)} (\nabla_X t) Y \\ & + \frac{n}{4(n+2)} \{(n+1)\epsilon_\alpha (dT)_\alpha(X, J_\alpha Y) - \epsilon_\beta (dT)_\beta(X, J_\beta Y) \\ & - \epsilon_\gamma (dT)_\gamma(X, J_\gamma Y)\}, \end{aligned} \tag{4.49}$$

where $(dT)_\alpha(X,Y) = \sum_{i=1}^{4n} \epsilon_i dT(X, Y, e_i, J_\alpha e_i)$.

Proof. Since the torsion is a 3-form, we have [20,25]

$$(\nabla^g_X T)(Y,Z,U) = (\nabla_X T)(Y,Z,U) + \frac{1}{2} \underset{XYZ}{\sigma} \{g(T(X,Y), T(Z,U))\}, \tag{4.50}$$

where $\underset{XYZ}{\sigma}$ denotes the cyclic sum of X, Y, Z.

The exterior derivative dT is given by

$$\begin{aligned} dT(X,Y,Z,U) \; = \; & \underset{XYZ}{\sigma} \{(\nabla_X T)(Y,Z,U) + g(T(X,Y), T(Z,U))\} \\ & - (\nabla_U T)(X,Y,Z) + \underset{XYZ}{\sigma} \{g(T(X,Y), T(Z,U))\}. \end{aligned} \tag{4.51}$$

The first Bianchi identity for ∇ states

$$\underset{XYZ}{\sigma} R(X,Y,Z,U) = \underset{XYZ}{\sigma} \{(\nabla_X T)(Y,Z,U) + g(T(X,Y), T(Z,U))\}. \tag{4.52}$$

We denote by B the Bianchi projector, i.e. $B(X,Y,Z,U) = \underset{XYZ}{\sigma} R(X,Y,Z,U)$.

The curvature R^g of the Levi-Civita connection is connected by R in the following way

$$\begin{aligned} R^g(X,Y,Z,U) \; = \; & R(X,Y,Z,U) - \frac{1}{2}(\nabla_X T)(Y,Z,U) + \frac{1}{2}(\nabla_Y T)(X,Z,U) \\ & - \frac{1}{2} g(T(X,Y), T(Z,U)) - \frac{1}{4} g(T(Y,Z), T(X,U)) - \frac{1}{4} g(T(Z,X), T(Y,U)). \end{aligned} \tag{4.53}$$

Define D by $D(X,Y,Z,U)=R(X,Y,Z,U)-R(Z,U,X,Y)$, we obtain from (4.53)

$$D(X,Y,Z,U)=\frac{1}{2}(\nabla_X T)(Y,Z,U)-\frac{1}{2}(\nabla_Y T)(X,Z,U)-\frac{1}{2}(\nabla_Z T)(U,X,Y)+\frac{1}{2}(\nabla_U T)(Z,X,Y), \quad (4.54)$$

since D^g of R^g is zero.

Using (4.46) and (4.52), we find the following relation between the Ricci tensor and the Ricci forms

$$\begin{aligned}\rho_\alpha(X,Y) &= -\frac{1}{2}\sum_{i=1}^{4n}(\epsilon_i(R(Y,e_i,X,J_\alpha e_i)+\epsilon_i R(e_i,X,Y,J_\alpha e_i)))+\frac{1}{2}\sum_{i=1}^{4n}\epsilon_i B(X,Y,e_i,J_\alpha e_i)\\ &= -\frac{1}{2}Ric(Y,J_\alpha X)+\frac{1}{2}Ric(X,J_\alpha Y)+\frac{1}{2}\sum_{i=1}^{4n}\epsilon_i B(X,Y,e_i,J_\alpha e_i)\\ &\quad +\frac{1}{2n}\left\{-\epsilon_\alpha\rho_\beta(J_\gamma Y,X)+\epsilon_\alpha\rho_\beta(J_\gamma X,Y)-\epsilon_\alpha\rho_\gamma(J_\beta X,Y)+\epsilon_\alpha\rho_\gamma(J_\beta Y,X)\right\}. \quad (4.55)\end{aligned}$$

On the other hand, using (4.46), we calculate

$$\begin{aligned}\sum_{i=1}^{4}\epsilon_i D(X,e_i,J_\alpha e_i,Y) &= \sum_{i=1}^{4n}\{\epsilon_i R(X,e_i,J_\alpha e_i,Y)+\epsilon_i R(Y,e_i,J_\alpha e_i X)\}\\ &= Ric(Y,J_\alpha X)+Ric(X,J_\alpha Y)\\ &\quad +\frac{1}{n}\left\{\epsilon_\alpha\rho_\beta(X,J_\gamma Y)+\epsilon_\alpha\rho_\beta(Y,J_\gamma X)-\epsilon_\alpha\rho_\gamma(Y,J_\beta X)-\epsilon_\alpha\rho_\gamma(X,J_\beta Y)\right\}. \quad (4.56)\end{aligned}$$

Combining (4.55) and (4.56), we derive

$$\begin{aligned}&n\epsilon_\alpha\rho_\alpha(X,J_\alpha Y)+\epsilon_\beta\rho_\beta(X,J_\beta Y)+\epsilon_\gamma\rho_\gamma(X,J_\gamma Y)\\ &\quad = nRic(X,Y)+\frac{n}{2}\epsilon_\alpha B_\alpha(X,J_\alpha Y)+\frac{n}{2}\epsilon_\alpha D_\alpha(X,J_\alpha Y), \quad (4.57)\end{aligned}$$

where the tensors B_α and D_α are defined by $B_\alpha(X,Y)=\sum_{i=1}^{4n}\epsilon_i B(X,Y,e_i,J_\alpha e_i)$ and $D_\alpha(X,Y)=\sum_{i=1}^{4n}\epsilon_i D(X,e_i,J_\alpha e_i,Y)$. Taking into account (4.54), we get the expression

$$D_\alpha(X,Y)=-(\nabla_X t)(J_\alpha Y)-(\nabla_Y t)(J_\alpha X) \quad \alpha=1,2,3. \quad (4.58)$$

To calculate $B_\alpha+D_\alpha$ we use (4.51) twice and (4.58). After some calculations, we derive

$$B_\alpha(X,Y)+D_\alpha(X,Y)=\frac{1}{2}\sum_{i=1}^{4n}\epsilon_i dT(X,Y,e_i,J_\alpha e_i)-2(\nabla_X t)(J_\alpha Y), \quad \alpha=1,2,3. \quad (4.59)$$

We substitute (4.59) into (4.57). Solving the system obtained, we obtain

$$(n-1)(\epsilon_\alpha\rho_\alpha(X,J_\alpha Y)-\epsilon_\beta\rho_\beta(X,J_\beta Y))=\frac{n}{2}(\epsilon_\alpha(dT)_\alpha(X,J_\alpha Y)-\epsilon_\beta(dT)_\beta(X,J_\beta Y)). \quad (4.60)$$

Finally, (4.57) and (4.59) imply (4.49). □

Theorem 4.4. *On a PQKT manifold* $(M^{4n},g,(J_\alpha)\in\mathbb{P})$ $(n>1)$ *the* $(2,0)+(0,2)$ *parts of the Ricci forms* ρ_α, ρ_β *with respect to* J_γ *coincide in the sense that the next identity holds*

$$\rho_\alpha(J_\beta X,J_\beta Y)+\epsilon_\beta\rho_\alpha(X,Y)+\epsilon_\gamma\rho_\gamma(J_\beta X,Y)+\epsilon_\gamma\rho_\gamma(X,J_\beta Y)=0. \quad (4.61)$$

Proof. We need the following

Lemma 4.5. *The tensors* $L_\alpha(X,Y)=\sum_{i=1}^{4n}\epsilon_i g(T(X,e_i),T(Y,J_\alpha e_i))$, *for* $\alpha=1,2,3$, *are related by:*

$$L_\alpha(J_\beta X,J_\beta Y)+\epsilon_\beta L_\alpha(X,Y)+\epsilon_\gamma L_\gamma(J_\beta X,Y)+\epsilon_\gamma L_\gamma(X,J_\beta Y)=0. \quad (4.62)$$

Proof. The formula (4.62) follows from equalities

$$
\begin{aligned}
L_\alpha(X,Y) &= \sum_{i=1}^{4n}\epsilon_i g(T(X,e_i),T(Y,J_\alpha e_i)) = -\sum_{i=1}^{4n}\epsilon_i g(T(X,J_\alpha e_i),T(Y,e_i))\\
&= \sum_{i,j=1}^{4n}\epsilon_i\epsilon_j T(X,e_i,e_j)T(e_j,Y,J_\alpha e_i) = \sum_{i,j=1}^{4n}\epsilon_i\epsilon_j T(X,e_i,e_j)g(T(e_j,Y),J_\alpha e_i)\\
&= \sum_{i,j=1}^{4n}\epsilon_i\epsilon_j T(X,e_j,e_i)g(e_i,J_\alpha T(Y,e_j)) = -\sum_{j=1}^{4n}\epsilon_j g(T(X,e_j),J_\alpha T(Y,e_j))
\end{aligned}
$$

and property (3.16). □

From the first Bianchi identity the next sequence of equalities follows

$$
\begin{aligned}
&2\rho_\alpha(J_\beta X,J_\beta Y)+\sum_{i=1}^{4n}\epsilon_i R(J_\beta Y,e_i,J_\beta X,J_\alpha e_i)+\sum_{i=1}^{4n}\epsilon_i R(e_i,J_\beta X,J_\beta Y,J_\alpha e_i) = 2\epsilon_\gamma(\nabla_{J_\beta X}t)J_\gamma Y\\
&\quad -2\epsilon_\gamma(\nabla_{J_\beta Y}t)J_\gamma X+\sum_{i=1}^{4n}\epsilon_i(\nabla_{e_i}T)(J_\beta X,J_\beta Y,J_\alpha e_i)\\
&\quad +\sum_{i=1}^{4n}\epsilon_i g(T(J_\beta X,J_\beta Y),T(e_i,J_\alpha e_i))-2L_\alpha(J_\beta X,J_\beta Y).\\
&2\epsilon_\beta\rho_\alpha(X,Y)+\sum_{i=1}^{4n}\epsilon_i\epsilon_\beta R(Y,e_i,X,J_\alpha e_i)+\sum_{i=1}^{4n}\epsilon_i\epsilon_\beta R(e_i,X,Y,J_\alpha e_i) = -2\epsilon_\beta(\nabla_X t)J_\alpha Y\\
&\quad +2\epsilon_\beta(\nabla_Y t)J_\alpha X+\sum_{i=1}^{4n}\epsilon_i\epsilon_\beta(\nabla_{e_i}T)(X,Y,J_\alpha e_i)+\sum_{i=1}^{4n}\epsilon_i\epsilon_\beta g(T(X,Y),T(e_i,J_\alpha e_i))-2\epsilon_\beta L_\alpha(X,Y).\\
&2\epsilon_\gamma\rho_\gamma(J_\beta X,Y)+\sum_{i=1}^{4n}\epsilon_i\epsilon_\gamma R(Y,e_i,J_\beta X,J_\gamma e_i)+\sum_{i=1}^{4n}\epsilon_i\epsilon_\gamma R(e_i,J_\beta X,Y,J_\gamma e_i) = -2\epsilon_\gamma(\nabla_{J_\beta X}t)J_\gamma Y\\
&\quad -2\epsilon_\beta(\nabla_Y t)J_\alpha X+\sum_{i=1}^{4n}\epsilon_i\epsilon_\gamma(\nabla_{e_i}T)(J_\beta X,Y,J_\gamma e_i)\\
&\quad +\sum_{i=1}^{4n}\epsilon_i\epsilon_\gamma g(T(J_\beta X,Y),T(e_i,J_\gamma e_i))-2\epsilon_\gamma L_\gamma(J_\beta X,Y).\\
&2\epsilon_\gamma\rho_\gamma(X,J_\beta Y)+\sum_{i=1}^{4n}\epsilon_i\epsilon_\gamma R(J_\beta Y,e_i,X,J_\gamma e_i)+\sum_{i=1}^{4n}\epsilon_i\epsilon_\gamma R(e_i,X,J_\beta Y,J_\gamma e_i) = 2\epsilon_\beta(\nabla_X t)J_\alpha Y\\
&\quad +2\epsilon_\gamma(\nabla_{J_\beta Y}t)J_\gamma X+\sum_{i=1}^{4n}\epsilon_i\epsilon_\gamma(\nabla_{e_i}T)(X,J_\beta Y,J_\gamma e_i)\\
&\quad +\sum_{i=1}^{4n}\epsilon_i\epsilon_\gamma g(T(X,J_\beta Y),T(e_i,J_\gamma e_i))-2\epsilon_\gamma L_\gamma(X,J_\beta Y).
\end{aligned}
$$

The sum of all these equalities, (4.46) and the fact that T is a $(1,2)+(2,1)$-form with respect to each J_α give

$$
\begin{aligned}
&\frac{2(n-1)}{n}\rho_\alpha(J_\beta X,J_\beta Y)+\frac{2(n-1)}{n}\epsilon_\beta\rho_\alpha(X,Y)+\frac{2(n-1)}{n}\epsilon_\gamma\rho_\gamma(J_\beta X,Y)+\frac{2(n-1)}{n}\epsilon_\gamma\rho_\gamma(X,J_\beta Y)\\
&\quad = -2L_\alpha(J_\beta X,J_\beta Y)-\epsilon_\beta 2L_\alpha(X,Y)-\epsilon_\gamma 2L_\gamma(J_\beta X,Y)-\epsilon_\gamma 2L_\gamma(X,J_\beta Y).
\end{aligned}
$$

From Lemma 4.5 and fact that $(n>1)$, we have (4.61). □

We easily derive from Theorem 4.4.

Corollary 4.6. *The* $(2,0)+(0,2)$ *parts of the 2-forms* $(dT)_\alpha$, $(dT)_\beta$ *with respect to* J_γ *coincide.*

Theorem 4.7. *On a* $4n$*-dimensional* $(n>1)$ *PQKT manifold the following formula holds*

$$\epsilon_\alpha\rho_\alpha(X,J_\alpha Y)+\epsilon_\alpha\rho_\alpha(J_\alpha X,Y)=-\frac{n}{n+1}(dt(X,Y)+\epsilon_\alpha dt(J_\alpha X,J_\alpha Y)). \tag{4.63}$$

In particular, ρ_α *is of type* $(1,1)$ *with respect to* J_α, $\alpha=1,2,3$, *if and only if* dt *is of type* $(1,1)$ *with respect to each* J_α, $\alpha=1,2,3$.

Proof. From the first Bianchi identity, formulas (3.16) and (4.46) it follows that

$$\begin{aligned}&2(\epsilon_\alpha\rho_\alpha(X,J_\alpha Y)+\epsilon_\alpha\rho_\alpha(J_\alpha X,Y))-\epsilon_\alpha(Ric(J_\alpha X,J_\alpha Y)-Ric(J_\alpha Y,J_\alpha X))-(Ric(X,Y)-Ric(Y,X))\\&\quad+\frac{1}{n}(\epsilon_\beta\rho_\beta(X,J_\beta Y)+\epsilon_\beta\rho_\beta(J_\beta X,Y)+\epsilon_\gamma\rho_\gamma(X,J_\gamma Y)+\epsilon_\gamma\rho_\gamma(J_\gamma X,Y)\\&\quad-\rho_\beta(J_\alpha X,J_\gamma Y)-\rho_\beta(J_\gamma X,J_\alpha Y)+\rho_\gamma(J_\alpha X,J_\beta Y)+\rho_\gamma(J_\beta X,J_\alpha Y))\\&=-2(dt(X,Y)+\epsilon_\alpha dt(J_\alpha X,J_\alpha Y))+\delta T(X,Y)+\epsilon_\alpha\delta T(J_\alpha X,J_\alpha Y).\end{aligned} \tag{4.64}$$

First, we substitute $X\to J_\alpha X$ into (4.61) to obtain

$$\epsilon_\gamma\rho_\alpha(J_\gamma X,J_\beta Y)+\epsilon_\beta\rho_\alpha(J_\alpha X,Y)+\rho_\gamma(J_\gamma X,Y)+\epsilon_\gamma\rho_\gamma(J_\alpha X,J_\beta Y)=0. \tag{4.65}$$

After that we substitute $Y\to J_\alpha Y$ into (4.61) to get

$$\epsilon_\gamma\rho_\alpha(J_\beta X,J_\gamma Y)+\epsilon_\beta\rho_\alpha(X,J_\alpha Y)+\rho_\gamma(X,J_\gamma Y)+\epsilon_\gamma\rho_\gamma(J_\beta X,J_\alpha Y)=0. \tag{4.66}$$

Summing up (4.65) and (4.66), we obtain

$$\begin{aligned}\epsilon_\alpha\rho_\alpha(X,J_\alpha Y)+\epsilon_\alpha\rho_\alpha(J_\alpha X,Y)&=\epsilon_\gamma\rho_\gamma(J_\gamma X,Y)+\epsilon_\gamma\rho_\gamma(X,J_\gamma Y)+\rho_\gamma(J_\alpha X,J_\beta Y)\\&\quad+\rho_\gamma(J_\beta X,J_\alpha Y)+(\rho_\alpha(J_\gamma X,J_\beta Y)+\rho_\alpha(J_\beta X,J_\gamma Y)).\end{aligned} \tag{4.67}$$

We perform the cyclic permutation $(\alpha,\beta,\gamma)\to(\beta,\gamma,\alpha)$ in (4.67) to obtain

$$\begin{aligned}\epsilon_\alpha\rho_\alpha(X,J_\alpha Y)+\epsilon_\alpha\rho_\alpha(J_\alpha X,Y)&=\epsilon_\beta\rho_\beta(J_\beta X,Y)+\epsilon_\beta\rho_\beta(X,J_\beta Y)-\rho_\beta(J_\alpha X,J_\gamma Y)\\&\quad-\rho_\beta(J_\gamma X,J_\alpha Y)-(\rho_\alpha(J_\gamma X,J_\beta Y)+\rho_\alpha(J_\beta X,J_\gamma Y)).\end{aligned} \tag{4.68}$$

Adding (4.67) to (4.68), we get

$$\begin{aligned}2(\epsilon_\alpha\rho_\alpha(X,J_\alpha Y)+\epsilon_\alpha\rho_\alpha(J_\alpha X,Y))&=\epsilon_\gamma\rho_\gamma(J_\gamma X,Y)+\epsilon_\gamma\rho_\gamma(X,J_\gamma Y)+\epsilon_\beta\rho_\beta(J_\beta X,Y)\\&\quad+\epsilon_\beta\rho_\beta(X,J_\beta Y)+\rho_\gamma(J_\alpha X,J_\beta Y)+\rho_\gamma(J_\beta X,J_\alpha Y)-\rho_\beta(J_\alpha X,J_\gamma Y)-\rho_\beta(J_\gamma X,J_\alpha Y).\end{aligned} \tag{4.69}$$

Now, equalities (4.64), (4.68), (4.69) and $Ric(X,Y)-Ric(Y,X)=-\delta T(X,Y)$ (see [25]) prove the assertion. □

Corollary 4.8. *On a* $4n$*-dimensional* $(n>1)$ *PQKT manifold the following formula holds*

$$\epsilon_\alpha\rho^g_\alpha(X,J_\alpha Y)+\epsilon_\alpha\rho^g_\alpha(J_\alpha X,Y)=-\frac{n-1}{2(n+1)}(dt(X,Y)+\epsilon_\alpha dt(J_\alpha X,J_\alpha Y)). \tag{4.70}$$

In particular, $\rho^\star_\alpha$ *is symmetric if and only if* dt *is of type* $(1,1)$ *with respect to each* J_α, $\alpha=1,2,3$.

Proof. We get from (4.53) that

$$\epsilon_\alpha\rho^g_\alpha(X,J_\alpha Y)+\epsilon_\alpha\rho^g_\alpha(J_\alpha X,Y)=\epsilon_\alpha\rho_\alpha(X,J_\alpha Y)+\epsilon_\alpha\rho_\alpha(J_\alpha X,Y)+\frac{1}{2}(dt(X,Y)+\epsilon_\alpha dt(J_\alpha X,J_\alpha Y)).$$

Now (4.70) is a consequence of (4.63). □

Proposition 4.9. *On a* $4n$*-dimensional* $(n>1)$ *PQKT manifold we have the equalities:*

$$Scal_{\alpha,\alpha}=Scal_{\beta,\beta}=Scal_{\gamma,\gamma},\quad Scal_{\alpha,\beta}=0,\quad Scal_\alpha=\frac{1}{2}(dt,\Phi_\alpha). \tag{4.71}$$

Proof. Using (4.73), we obtain

$$\frac{2(n-1)}{n}(\rho_\alpha(X,J_\alpha Y)-\rho_\beta(X,J_\beta Y))=((dT)_\alpha(X,J_\alpha Y)-(dT)_\beta(X,J_\beta Y)); \tag{4.72}$$

$$\begin{aligned}(n-1)\epsilon_\alpha\rho_\alpha(X,J_\alpha Y) = {} & \frac{n(n-1)}{n+2}Ric(X,Y)-\frac{n(n-1)}{n+2}(\nabla_X t)Y+\frac{n}{4(n+2)} \\ & \times\left\{(n+1)\epsilon_\alpha(dT)_\alpha(X,J_\alpha Y)-\epsilon_\beta(dT)_\beta(X,J_\beta Y)-\epsilon_\gamma(dT)_\gamma(X,J_\gamma Y)\right\}.\end{aligned} \tag{4.73}$$

Take the appropriate trace in (4.72), to get $Scal_{\alpha,\alpha}=Scal_{\beta,\beta}, Scal_{\alpha,\beta}=0$. The last equality in (4.71) is a direct consequence of $Scal_{\alpha,\beta}=0$ and (4.73). □

Definition. The three coinciding traces of the Ricci forms on a $4n$-dimensional PQKT manifold ($n>1$) give a well defined global function. We call this function the **paraquaternionic scalar curvature of the PQKT connection** and denote it by $Scal_{\mathbb{P}}:=Scal_{\alpha,\alpha}$.

Proposition 4.10. *On a* $4n$*-dimensional* ($n>1$) *PQKT manifold we have*

$$Scal^g_\alpha=Scal^g_\beta=Scal^g_\gamma=Scal_{\mathbb{P}}-\delta t+\|t\|^2-\frac{1}{12}\|T\|^2,\ -\epsilon_\gamma Scal^g_{\alpha,\beta}=Scal_\gamma=\frac{1}{2}(dt,\Phi_\gamma). \tag{4.74}$$

Proof. The curvature R^g of the Levi-Civita connection is related to R via (4.53). Taking the traces in (4.53) and using (3.19), we obtain

$$\begin{aligned}\epsilon_\alpha\rho^g_\alpha(X,J_\alpha Y) = {} & \epsilon_\alpha\rho_\alpha(X,J_\alpha Y)+\frac{1}{2}(\nabla_X t)Y-\epsilon_\alpha\frac{1}{2}(\nabla_{J_\alpha Y}t)J_\alpha X \\ & +\frac{1}{2}\epsilon_\alpha t(J_\alpha T(X,J_\alpha Y))+\frac{1}{4}\sum_{i=1}^{4n}\epsilon_i\epsilon_\alpha g\left(T(X,e_i),T(J_\alpha Y,J_\alpha e_i)\right).\end{aligned} \tag{4.75}$$

To finish, take the appropriate traces in (4.75) and apply Proposition 4.9. □

Definition. The three coinciding traces of the Riemannian Ricci forms on a $4n$-dimensional PQKT manifold ($n>1$) give a well defined global function. We call this function the **paraquaternionic** $*$**-scalar curvature** and denote it by $Scal^g_{\mathbb{P}}:=Scal^g_{\alpha,}$.

Proposition 4.11. *On a* $4n$*-dimensional* ($n>1$) *PQKT manifold* $(M,g,\mathbb{P})$ *the scalar curvatures are related by*

$$\begin{aligned}Scal^g &= \frac{n+2}{n}Scal_{\mathbb{P}}-3\delta t+2\|t\|^2-\frac{1}{12}\|T\|^2,\\ Scal^g_{\mathbb{P}} &= Scal_Q-\delta t+\|t\|^2-\frac{1}{12}\|T\|^2,\\ Scal &= \frac{n+2}{n}Scal_{\mathbb{P}}-3\delta t+2\|t\|^2-\frac{1}{3}\|T\|^2.\end{aligned}$$

Proof. We derive from (4.53) that

$$\begin{aligned}Ric^g(X,Y) &= Ric(X,Y)+\frac{1}{2}\delta T(X,Y)+\frac{1}{4}\sum_{i=1}^{2n}g\left(T(X,e_i),T(Y,e_i)\right),\\ Scal^g &= Scal+\frac{1}{4}\|T\|^2.\end{aligned} \tag{4.76}$$

Take the trace in (4.73) to get the first equality of the proposition. The second equality is already proved in Proposition 4.10. The last one is a consequence of (4.76) and the already proven first equality in the proposition. □

5. PQKT manifolds with parallel torsion and homogeneous PQKT structures

Let $(G/K, g)$ be a reductive (locally) homogeneous pseudo-Riemannian manifold. The canonical connection ∇ is characterized by the properties $\nabla g = \nabla T = \nabla R = 0$ [32]. A homogeneous paraquaternionic Hermitian manifold (resp. homogeneous hyper-para-Hermitian) manifold $(G/K, g, \mathbb{P})$ is a homogeneous pseudo-Riemannian manifold with an invariant paraquaternionic Hermitian subbundle $\mathbb{P}$ (resp. three invariant anti-commuting (para)complex structures). This means that the bundle $\mathbb{P}$ (resp. each of the (para)complex structures) is parallel with respect to the canonical connection ∇. The torsion of ∇ is totally skew symmetric if and only if the homogeneous pseudo-Riemannian manifold is naturally reductive. Homogeneous PQKT (resp. HPKT) manifolds are homogeneous paraquaternionic Hermitian (resp. homogeneous hyper-para-Hermitian) manifolds which are naturally reductive.

We show that there are no homogeneous PQKT manifolds with torsion 4-form dT of type (2, 2) with respect to each J_α in dimensions greater than four. First, we prove the following technical result

Proposition 5.1. *Let $(M, g, (J_\alpha), \nabla)$ be a $4n$-dimensional $(n > 1)$ PQKT manifold with 4-form dT of type* (2, 2) *with respect to each J_α, $\alpha = 1, 2, 3$. Suppose that the torsion is parallel with respect to the PQKT connection. Then the Ricci forms ρ_α are given by*

$$\epsilon_\alpha \rho_\alpha(X, J_\alpha Y) = \lambda g(X, Y), \quad \alpha = 1, 2, 3, \tag{5.77}$$

where λ is a smooth function on M.

Proof. Let the torsion be parallel, i.e. $\nabla T = 0$. This implies that the Ricci tensor is symmetric [20]. The equalities (4.51) and (4.52) lead to

$$B(X, Y, Z, U) = \underset{XYZ}{\sigma}\{g(T(X, Y), T(Z, U))\} = \frac{1}{2} dT(X, Y, Z, U). \tag{5.78}$$

We get $D = 0$ from (4.54).

Suppose now that the 4-form dT is of type (2, 2) with respect to each J_α, $\alpha = 1, 2, 3$. Then it satisfies the equalities

$$-\epsilon_\alpha dT(X, Y, Z, U) = dT(J_\alpha X, J_\alpha Y, Z, U) + dT(J_\alpha X, Y, J_\alpha Z, U) + dT(X, J_\alpha Y, J_\alpha Z, U). \tag{5.79}$$

Arguments similar to those we used in the proof of Proposition 3.1, but applying (5.79) instead of (3.16), yield

Lemma 5.2. *On a PQKT manifold with 4-form dT of type* (2, 2) *with respect to each J_α, $\alpha = 1, 2, 3$, the following equalities hold:*

$$(dT)_1(X, J_1 Y) = (dT)_2(X, J_2 Y) = -(dT)_3(X, J_3 Y), \tag{5.80}$$

$$(dT)_\alpha(X, J_\alpha Y) = -(dT)_\alpha(J_\alpha X, Y), \quad \alpha = 1, 2, 3. \tag{5.81}$$

We substitute (5.80), (5.78) and $D = 0$ into (4.73) to get

$$\rho_1(X, J_1 Y) = \rho_2(X, J_2 Y) = -\rho_3(X, J_3 Y), \tag{5.82}$$

$$\epsilon_\alpha \rho_\alpha(X, J_\alpha Y) = \frac{n}{n+2} Ric(X, Y) + \frac{n}{4(n+2)} \epsilon_\alpha (dT)_\alpha(X, J_\alpha Y), \quad \alpha = 1, 2, 3. \tag{5.83}$$

The equality (5.81) shows that the 2-form dT_α is a (1, 1)-form with respect to J_α. Hence, the dT_α is (1, 1)-form with respect to each J_α, $\alpha = 1, 2, 3$, because of (5.80). Since the Ricci tensor Ric is symmetric, (5.83) shows that the Ricci tensor Ric satisfies $Ric(J_\alpha X, J_\alpha Y) = -\epsilon_\alpha Ric(X, Y)$, $\alpha = 1, 2, 3$, for each J_α and the Ricci forms ρ_α, $\alpha = 1, 2, 3$, are (1, 1)-forms with respect to all J_α, $\alpha = 1, 2, 3$. Taking into account (4.46), we obtain

$$\begin{aligned} &-\epsilon_\alpha R(X, J_\alpha X, Z, J_\alpha Z) + R(X, J_\alpha X, J_\beta Z, J_\gamma Z) + R(J_\beta X, J_\gamma X, Z, J_\alpha Z) \\ &\quad - \epsilon_\alpha R(J_\beta X, J_\gamma X, J_\beta Z, J_\gamma Z) = \frac{1}{n}\left(-\epsilon_\alpha \rho_\alpha(X, J_\alpha X) + \rho_\alpha(J_\beta X, J_\gamma X)\right) g(Z, Z) \\ &\quad = -\frac{2}{n} \epsilon_\alpha \rho_\alpha(X, J_\alpha X) g(Z, Z) \end{aligned} \tag{5.84}$$

where the last equality of (5.84) is a consequence of the following identity

$$\rho_\alpha(J_\beta X, J_\gamma X) = \epsilon_\beta \rho_\beta(J_\beta X, X) = -\epsilon_\alpha \rho_\alpha(X, J_\alpha X).$$

The left-hand side of (5.84) is symmetric with respect to the vectors X, Z because $D = 0$. Hence, $\rho_\alpha(X, J_\alpha X) g(Z, Z) = \rho_\alpha(Z, J_\alpha Z) g(X, X)$, $\alpha = 1, 2, 3$. The last equality together with (5.82) implies (5.77). □

Theorem 5.3. *Let $(M, g, (J_\alpha))$ be a $4n$-dimensional $(n > 1)$ PQKT manifold with 4-form dT of type $(2, 2)$ with respect to each $J_\alpha, \alpha = 1, 2, 3$. Suppose that the torsion is parallel with respect to the PQKT connection. Then $(M, g, (J_\alpha))$ is either an HPKT manifold with parallel torsion or a PQK manifold.*

Proof. We apply Proposition 5.1. If the function $\lambda = 0$ then $\rho_\alpha = 0, \alpha = 1, 2, 3$, by (5.77) and Proposition 4.2 implies that the PQKT manifold is actually an HPKT manifold.

Let $\lambda \neq 0$. The condition (5.77) determines the torsion completely. We proceed by involving (4.47) in the computations. We calculate, using (2.1) and (5.77), that

$$(\nabla_Z \rho_\alpha)(X, Y) = \lambda \left\{\omega_\beta(Z) F_\gamma(X, Y) + \epsilon_\gamma \omega_\gamma(Z) F_\beta(X, Y)\right\} + d\lambda(Z) F_\alpha(X, Y). \tag{5.85}$$

Applying the operator d to (4.46), we get taking into account (5.77) that

$$d\rho_\alpha = \lambda(\epsilon_\gamma F_\beta \wedge \omega_\gamma + \omega_\beta \wedge F_\gamma). \tag{5.86}$$

On the other hand, we have

$$d\rho_\alpha(X, Y, Z) = \underset{XYZ}{\sigma} \{(\nabla_Z \rho_\alpha)(X, Y) + \lambda(T(X, Y, J_\alpha Z))\}, \quad \alpha = 1, 2, 3. \tag{5.87}$$

Comparing the left-hand sides of (5.86) and (5.87) and using (5.85), we derive

$$\lambda \underset{XYZ}{\sigma} \{g(T(X, Y), J_\alpha Z)\} = -d\lambda \wedge F_\alpha(X, Y, Z), \quad \alpha = 1, 2, 3.$$

The last equality implies $\lambda T = \epsilon_\alpha J_\alpha d\lambda \wedge F_\alpha, \quad \alpha = 1, 2, 3$. If λ is a non-zero constant then $T = 0$. If λ is not a constant then there exists a point $p \in M$ and a neighbourhood V_p of p such that $\lambda|_{V_p} \neq 0$. Then

$$T = \epsilon_\alpha J_\alpha d \ln \lambda \wedge F_\alpha, \quad \alpha = 1, 2, 3. \tag{5.88}$$

We take the trace in (5.88) to obtain

$$4(n - 1) J_\alpha d \ln \lambda = 0, \quad \alpha = 1, 2, 3. \tag{5.89}$$

Eq. (5.89) forces $d\lambda = 0$ since $n > 1$ and consequently $T = 0$ by (5.88). Hence, the PQKT space is a PQK manifold which completes the proof. □

On a locally homogeneous PQKT manifold the torsion and curvature are parallel and Theorem 5.3 leads to the following

Theorem 5.4. *A (locally) homogeneous $4n$-dimensional $(n > 1)$ PQKT manifold with torsion 4-form dT of type $(2, 2)$ is either a (locally) homogeneous HPKT space or a (locally) symmetric PQK space.*

Theorem 5.4 shows that there are no homogeneous (proper) PQKT manifolds with torsion 4-form of type (2, 2) in dimensions greater than four.

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